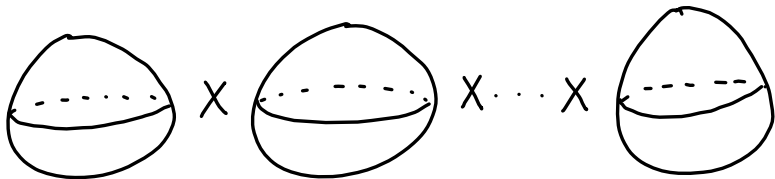


The Moment/SOS Hierarchy on the Product of Spheres



Sami Halaseh

Joint work with Victor Magron
+ Mateusz Skomra

LAAS CNRS

Konstanz University

Part of the TENORS Network

Polynomial Optimization Problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S^{n_1-1} \times \dots \times S^{n_m-1} \end{array}$$

Polynomial Optimization Problem

$$\min f(x)$$

$$\text{s.t. } x \in S^{n_1-1} \times \dots \times S^{n_m-1}$$

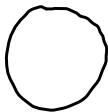
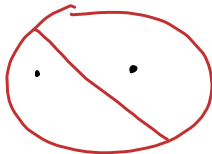
$$S^{n_i-1} = \left\{ x \in \mathbb{R}^{n_i} \mid 1 - \|x\|^2 = 0 \right\}$$

$$n_i \geq 2$$

Polynomial Optimization Problem

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$$\text{s.t. } x \in S^{n_1-1} \times \dots \times S^{n_m-1}$$

Def w/ $x = (x_1, \dots, x_n)$ $x_i = (x_{i1}, \dots, x_{in_i}) \in \mathbb{R}^{n_i}$

f **multihomogeneous** of multi degree $(d_1, \dots, d_m) \in \mathbb{N}_{\geq 1}^m$
if $\deg_{x_i} f = d_i$ for all $i \in [m]$

Polynomial Optimization Problem

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if $\deg_{x_i} f = d_i$ for all $i \in [m]$

eg

$$\begin{aligned} n_1 &= 2 \\ n_2 &= 3 \end{aligned}$$

$$x_{11}^2 x_{22} + x_{11} x_{12} x_{23} \rightarrow \text{mdeg} = (2, 1)$$

$$x_{11}^3 + x_{12} x_{21}^2 \rightarrow \text{homogeneous but } \underline{\text{not}} \text{ multihomogeneous.}$$

Polynomial Optimization Problem

$$\min f(x)$$

$$\text{s.t. } x \in S^{n_1-1} \cdots x \in S^{n_m-1}$$

$$\bullet S^{n_i-1} = \{x \in \mathbb{R}^{n_i-1} \mid \|x_i\|^2 = 1\}$$

$\bullet f$ multihomogeneous

Polynomial Optimization Problem

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Can replace
with $\mathbb{C}$!

Polynomial Optimization Problem

$$\min f(x)$$

$$\text{s.t. } x \in S^{n_1-1} \cdots x S^{n_m-1}$$

$$\bullet S^{n_i-1} = \{x \in \mathbb{R}^{n_i-1} \mid \|x_i\|^2 = 1\} \rightsquigarrow \bullet S_{\mathbb{C}}^{n-1} = \{z \in \mathbb{C}^n \mid \|z\|^2 = 1\}$$

• f multihomogeneous

Polynomial Optimization Problem

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Polynomial Optimization Problem

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- f multihomogeneous  • f real valued and multihomogeneous

Polynomial Optimization Problem

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$\mathbb{R}$

$$\bullet S^{n_i-1} = \{x \in \mathbb{R}^{n_i-1} \mid \|x_i\|^2 = 1\}$$

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$\mathbb{C}$

$$\bullet S_{\mathbb{C}}^{n-1} = \{z \in \mathbb{C}^n \mid \|z\|^2 = 1\} \cong S^{2n-1}$$

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Polynomial Optimization Problem

$$\textcircled{\text{pop}} \quad \min f(x) \\ \text{s.t. } x \in S^{n_1-1} \cdots x \in S^{n_m-1}$$

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$$\bullet S^{n_i-1} = \{x \in \mathbb{R}^{n_i-1} \mid \|x_i\|^2 = 1\}$$

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Generalized Moment Problem

Generalized Moment Problem

$$\begin{array}{ll} \text{pop} & \min f(x) \\ & \text{s.t. } x \in S^{n_1-1} \cdots x \in S^{n_n-1} \end{array}$$

Generalized Moment Problem

pop $\min f(x)$
s.t. $x \in S^{n_1-1} \times \dots \times S^{n_m-1}$

$\longleftrightarrow$

$\inf \int f d\mu$
s.t. $\int d\mu = 1$
 $\mu \in \mathcal{M}_+(S^{n_1-1} \times \dots \times S^{n_m-1})$

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nonnegative
Borel measures
supported on

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$\inf \int f d\mu$

$$\begin{array}{l} \text{s.t. } \int h_i d\mu = b_i \quad i=1, \dots, l \\ \mu \in \mathcal{M}_+(S^{n_1-1} \times \dots \times S^{n_m-1}) \end{array}$$

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Polynomial $\rightarrow$ h_i

b_i $\rightarrow$ Real number

Generalized Moment Problem

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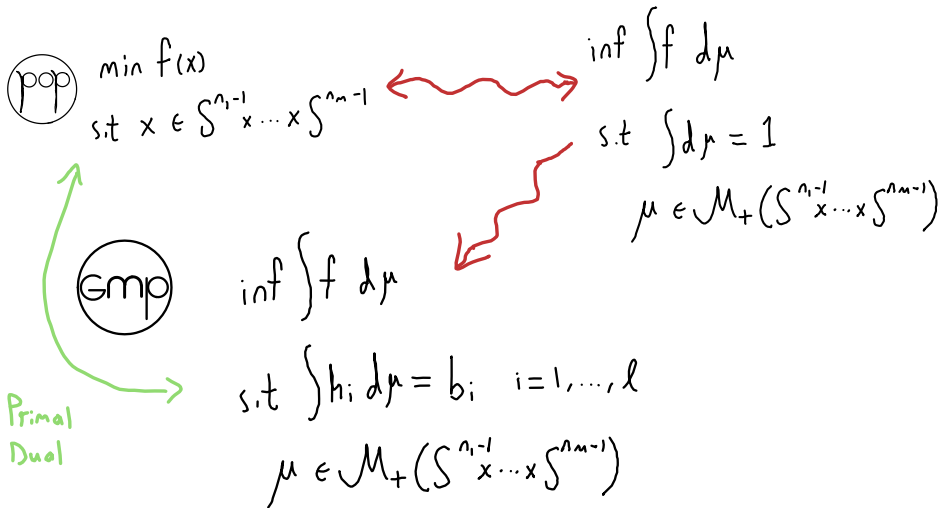
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Primal
Dual

The diagram illustrates the relationship between three optimization problems. A green arrow points from the 'Primal' label to the first problem. A red wavy arrow points from the first problem to the second problem. A red zigzag arrow points from the second problem to the third problem.

Generalized Moment Problem



Why use pop Gmp?

Why use pop Gmp? → Modeling Power!

Why use pop Gmp ? $\rightarrow$ Modeling Power!

pop

Why use pop Gmp ? $\rightarrow$ Modeling Power!

pop

$\hookrightarrow$ Best rank one approx tensor [NW'14]

Why use pop Gmp ? $\rightarrow$ Modeling Power!

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$\hookrightarrow$ SOS convexity [Las'08] [ABP'25]

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$\hookrightarrow$ DPS Hierarchy [DPS'04] [GLS'21]

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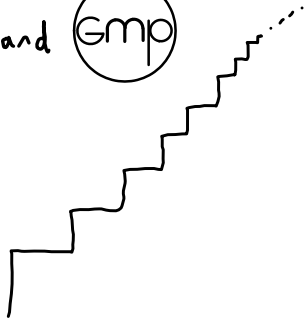
Gmp

$\hookrightarrow$ DPS Hierarchy [DPS'04] [GLS'21]

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Moment / SOS

Our tool for solving pop and Gmp



Moment / SOS

Generally

$$p_{\min} = \inf_{x \in K} P(x)$$

Moment / SOS

Generally $\rightarrow$ Polynomial

$$P_{\min} = \inf_{x \in K} P(x)$$

$\rightarrow = \{x \in \mathbb{R}^n \mid s_i(x) = 0, g_j(x) \geq 0\}$

semialgebraic set

Moment / SOS

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Moment / SOS

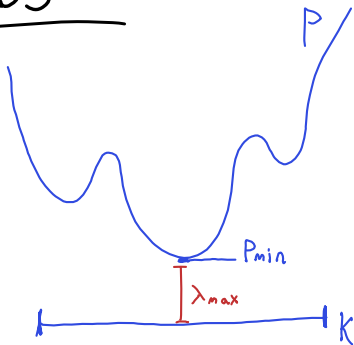
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$$P_{\min} = \inf_{x \in K} P(x) \longleftrightarrow P_{\min} = \sup_{\lambda} \lambda \quad \text{s.t. } P - \lambda \geq 0 \text{ on } K$$

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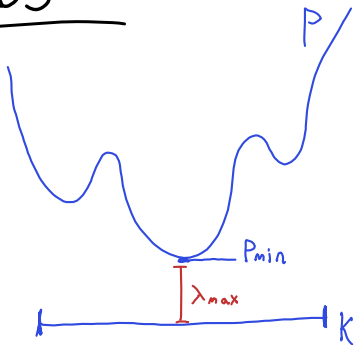


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Strengthen for
 $K = \{x \in \mathbb{R}^n \mid s_i(x) = 0, g_j(x) \geq 0\}$



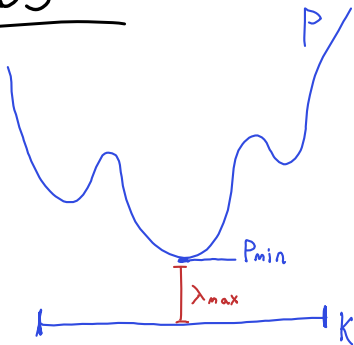
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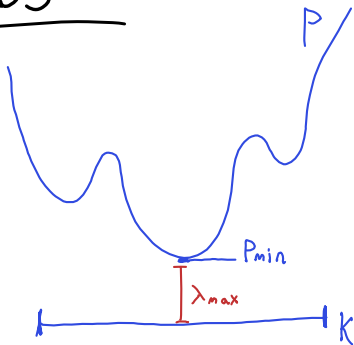
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Sum of Squares

so always ≥ 0

Arbitrary
polynomial



Moment / SOS

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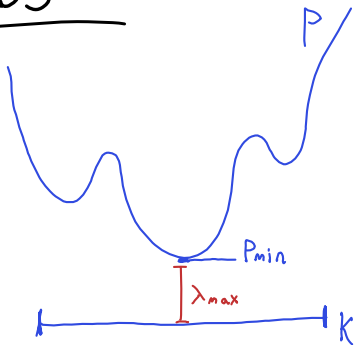
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Arbitrary
 polynomial

≥ 0

By def
 of K



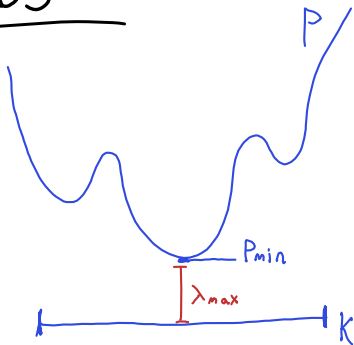
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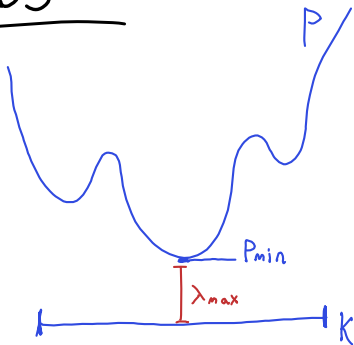
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Truncate degree
 $\downarrow$

$$P_{\text{SOS}}^t = \sup_{s.t. \ P - \lambda = \sigma_0 + \sum_j \sigma_j g_j + \sum_i q_i s_i} \lambda$$

and $\deg \sigma_0, \deg \sigma_j g_j, \deg q_i s_i \leq 2t$



Moment/SOS

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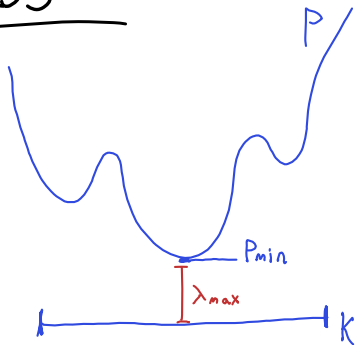
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We have

$$P_{\text{SOS}}' \leq \dots \leq P_{\text{SOS}}^t \leq P_{\text{SOS}} \leq P_{\min}$$

Moment / SOS

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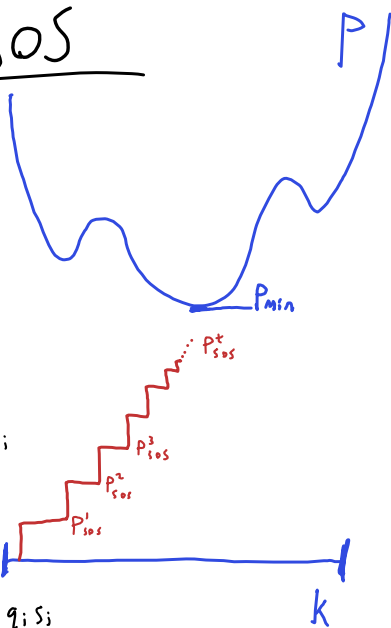
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Moment/SOS

P

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$$K = \{x \in \mathbb{R}^n \mid \dots\}$$

$$P_{\text{sos}} = \sup_{\lambda} \lambda \quad \text{s.t. } P - \lambda = \sigma_0$$

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The relaxation of  gives the dual semidefinite program

So $P_{\text{sos}}^t \leq P_{\text{mom}}^t \leq P_{\min}$

k

Moment/SOS

P

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The relaxation of  gives the dual semidefinite program

$$\text{So } P_{\text{SOS}}^t \leq P_{\text{mom}}^t \leq P_{\min}$$

sup ↑ inf

k

Why use Moment/SOS?

Why use Moment/SOS?

Nice properties
when applied
to pop Gmp

Why use Moment/SOS?

pop

Nice properties
when applied
to pop gmp

Why use Moment/SOS?



↳ Finite convergence (generically) [HMS'25]

Nice properties
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to



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↳ Good convergence rate ($O(1/t^2)$) [Mag'25]

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↳ Symmetries and algebraic properties to exploit

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to  

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- ↳ Good Convergence rate ($O(1/t^2)$) [Mag'25]
- ↳ Symmetries and algebraic properties to exploit
- ↳ Strong numerical results with nice solvers (TSSOS, MomentOpt,...)

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- ↳ Duality attainment + strict feasibility [HMS '26]

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- ↳ Good Convergence rate ($O(1/t^2)$) [Mag'25]
- ↳ Symmetries and algebraic properties to exploit
- ↳ Strong numerical results with nice solvers (TSSOS, MomentOpt,...)



- ↳ Duality attainment + strict feasibility [HMS '26]

Duality Attainment

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Recall

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For both of the above

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$$K = \left\{ (s, h, d, r)_{i=1, \dots, m} \mid \mu \in \mathcal{M}_+(S^{n-1} \times \dots \times S^{n-1}) \right\}$$

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assume $b \in \text{relint}(K)$

Proof Idea and Implications

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Two Proofs

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- infinite dim $\rightarrow$ existence of optimality certificates
- finite dim $\rightarrow$ stability of Primal/dual interior point solvers

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- Model many problems in tensor optimization
and quantum information theory
- Moment/SOS hierarchy offers tractable lower bounds
or global optima