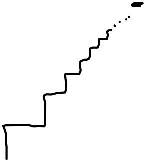
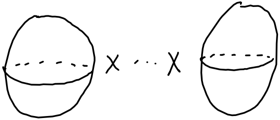
Finite Convergence of the Moment/SOS Hierarchy on the Product of Spheres





Sami Halaseh
LAAS-CNRS Toulouse
Konstanz University

Joint w/ Victor Magron
+ Mateusz Skomra

Part of the TENORS Network

Polynomial Optimization Problem

$$\min f(x)$$
$$\text{s.t. } x \in S^{n_1 - 1} \times \cdots \times S^{n_m - 1}$$

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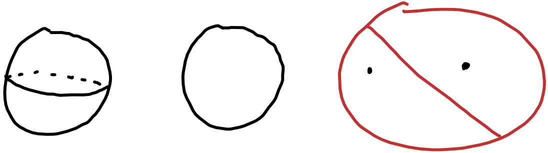

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Def $x = (x_1, \dots, x_n)$ $x_i = (x_{i1}, \dots, x_{in_i}) \in \mathbb{R}^{n_i}$

f multihomogeneous of multidegree $(d_1, \dots, d_m) \in \mathbb{N}^m_{\geq 1}$

if $\deg_{x_i} f = d_i$ for all $i \in [m]$

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eg $\quad n_1 = 2$, $n_2 = 3$

$x_{11}^2 x_{22} + x_{11} x_{12} x_{23} \longrightarrow \text{mdeg} = (2,1)$

$x_{11}^3 + x_{12} x_{21}^2 \longrightarrow$ homogeneous but not multihomogeneous.

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Vector Space of multihomogeneous polynomials of multidegree $(d_1,\ldots,d_m)$

Polynomial Optimization Problem

(POP) $\min f(x)$
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   - ↳ Best rank one approx tensor [NW'14]

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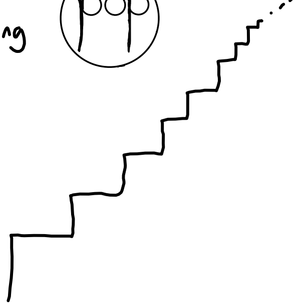
*

Moment/SoS

Our tool for solving pop



Moment/SOS

Generally

$$\begin{aligned} &\min P(x) \\ &\text{s.t } x \in K \end{aligned}$$

Moment/ SOS

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Polynomial

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semialgebraic, ie

$$K = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ g_j(x) \geq 0\}$$

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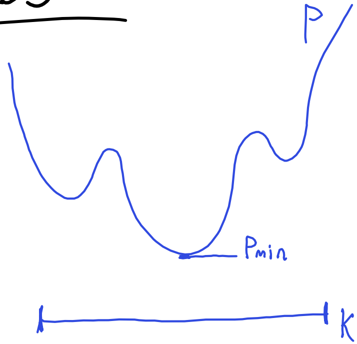

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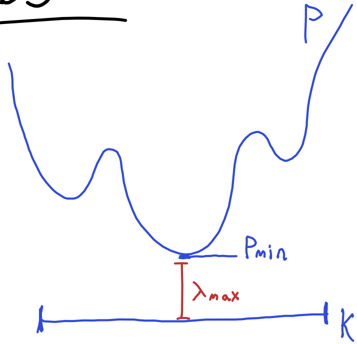

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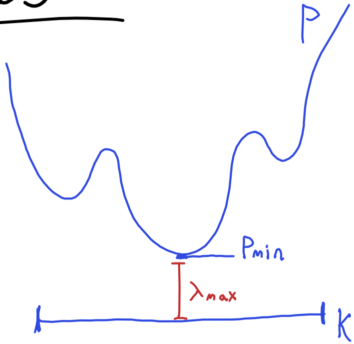

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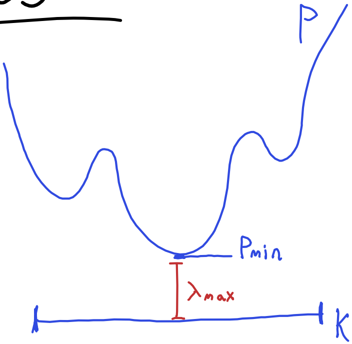

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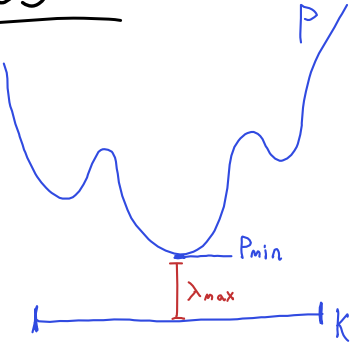
Sum of Squares

so always ≥ 0

Arbitrary polynomial



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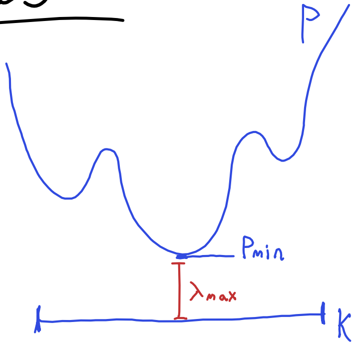
σ_0, σ_j: Sum of Squares so always ≥ 0

q_i: Arbitrary polynomial

≥ 0: By def of K



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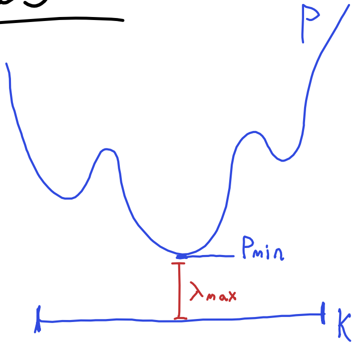

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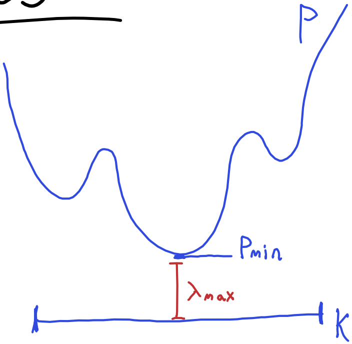

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and $\deg \sigma_0,\ \deg \sigma_j g_j,\ \deg q_i h_i \leq 2t$

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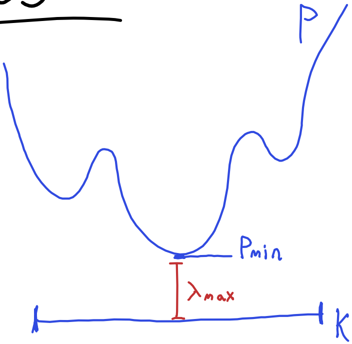

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For us "max = sup" but not in general

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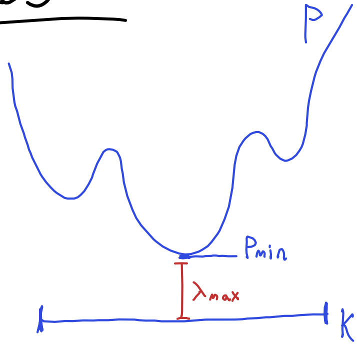

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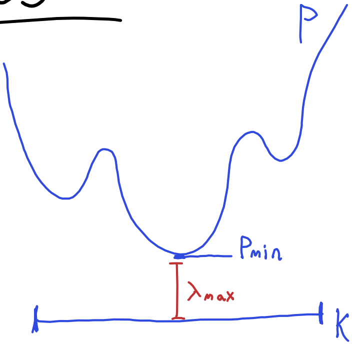

We have

$$P^1_{sos} \leq \dots \leq P^t_{sos} \leq P_{sos} \leq P_{min}$$

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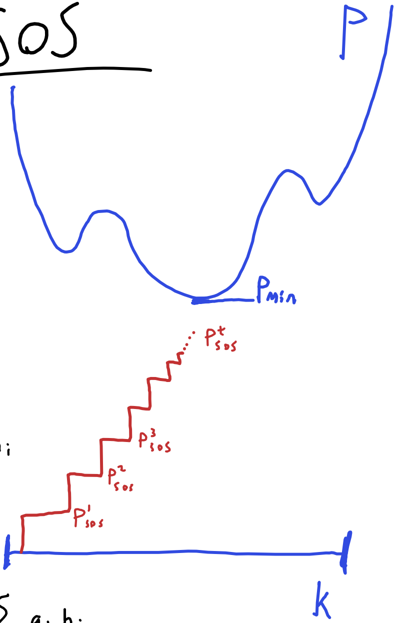

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- <u>Putinar</u>

  Assuming Archimedeanness we have $p_{sos}^t \xrightarrow{t \to \infty} p_{min}$

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↳ Description of K has $g_1 = R - \|x\|^2$ for some $R > 0$

Moment/SOS Convergence

$$\boxed{p_{sos}^{t} \longrightarrow p_{min} \, ?}$$

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Assuming Archimedeanness we have $p_{sos}^{t} \xrightarrow{t \to \infty} p_{min}$ "Asymptotic" convergence

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- **Nie (Scheiderer, Marshall)**

  Assuming Arch. and local optimality conditions on all global minimizers

  we have $\exists t_0 \in \mathbb{N}$ s.t. $p^{t_0}_{sos} = p_{min}$

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  Like first and second derivative test but using Lagrangian multipliers "gradient is zero" + "Hessian positive definite"

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  we have $\exists t_0 \in \mathbb{N}$ s.t $p_{sos}^{t_0} = p_{min}$ $\rightarrow$ finite convergence

Moment/SOS for POP

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- Case of one sphere

Moment/SOS for (POP)

- Case of one sphere

  Thm [Huang '23]

  For generic homogeneous $f \in \mathbb{R}[x]_{=d}$ and $m=1$
  the mom/sos hierarchy applied to (POP) has finite convergence

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  Thm [Huang '23]

  Number of spheres

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Moment/SOS for POP

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For generic homogeneous $f \in \mathbb{R}[x]_{=d}$ and $m=1$
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↳ Proof uses
- Projective Elimination Theory
- Nie's Thm (local opt conditions)

Moment/ SOS for (pop)

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  the mom/sos hierarchy applied to (pop) has finite convergence

Moment/SOS for POP

- Case of one sphere

  Thm [Huang '23]

  For generic homogeneous $f \in \mathbb{R}[x]_{=d}$ and $m=1$ the mom/sos hierarchy applied to POP has finite convergence

- Any number of Spheres

  Thm [HMS '25]

  For generic multihomogeneous $f \in \mathbb{R}[x]_{=(d_1 \dots d_m)}$ the mom/sos hierarchy applied to POP has finite convergence

Proof Sketch

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- Sard's Theorem

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- **Sard's Theorem**

  $U \subset \mathbb{R}^l$ open

  $\varphi : U \longrightarrow \mathbb{R}^n$ smooth

  Then the set of critical values of φ in $\mathbb{R}^n$ has Lebesgue measure zero

Proof Sketch

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Points in image where Jacobian of f does not have full rank

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Recall Nie's thm needed at all global minimizers!

Proof Sketch

- **Sard's Theorem**

  $U \subset \mathbb{R}^l$ open
  $\varphi : U \longrightarrow \mathbb{R}^n$ smooth

  Then the set of critical values of φ in $\mathbb{R}^n$ has Lebesgue measure zero

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Fixed mdeg $\mathbb{R}[X]_{=(d_1,\dots d_m)}$ is a vector space parametrized by coeff's

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Show that outside some Lebesgue msr zero set all polynomials are Morse

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Thm If map from $\mathcal{C}$ to tangent space surjective

then $\forall c \in \mathcal{C} \setminus Z$ f_c Morse, Z Leb msr zero

Application of pop

Best rank one approximating tensor

Best rank 1 tensor

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Def

- $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ order m tensor

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$\hookrightarrow$ Very hard to find and can be high

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Like Frobenius Norm

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pop with $d_1, \dots, d_m = 1$

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Cor Given a generic tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times \dots \times n_m}$ the best rank one approximating tensor can be found in finitely many steps of the moment/SoS hierarchy

How good is rank one?

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Matrix Example (SVD)

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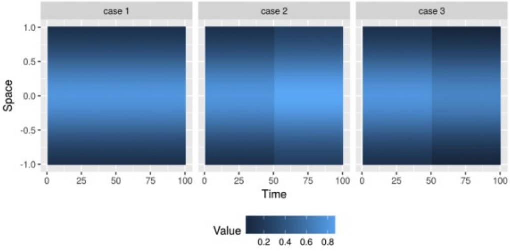
Matrix Example (SVD)

credit: Samir Beall - Wikipedia

Tensor Example

Tensor Example



credit: Alexej's blog on Rbloggers

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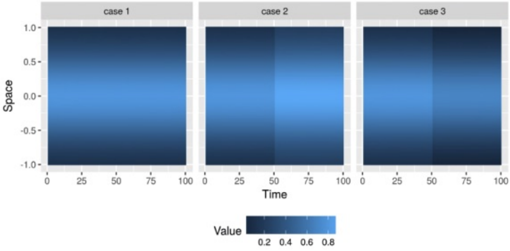

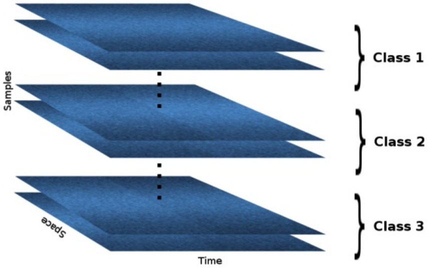

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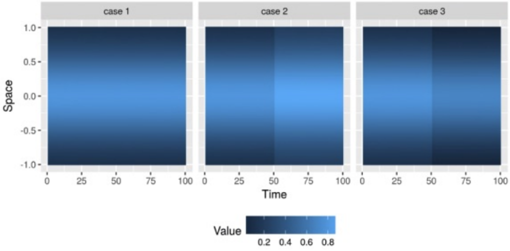

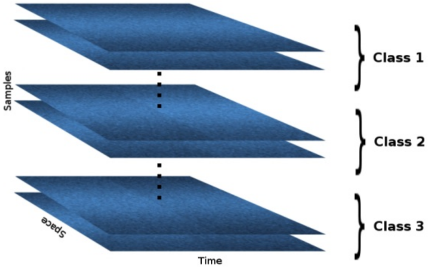

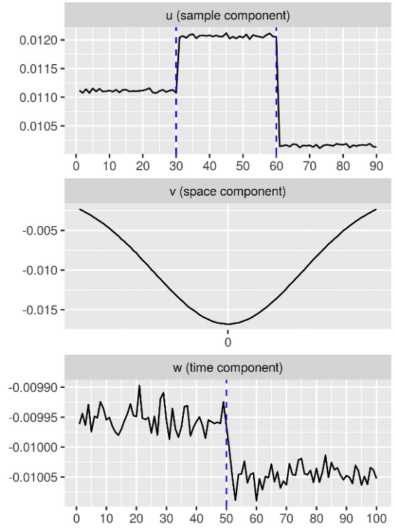

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s.t $x \in S^{n_1-1} \times \cdots \times S^{n_m-1}$
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